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Abstract. The gauge group as the algebraic closure of two finite-dimensional groups is investigated. The Yang–Mills theory in the above context is considered.

1. Introduction

In the works of Ogievetsky *et al* (Ogievetsky 1973, Borisov and Ogievetski 1974) an interesting method has been presented for obtaining the theory which is invariant under the general covariant group and equivalent to Einstein's theory. This method is based on the fact that the Lie algebra of the general covariant group is the algebraic closure of the conformal and affine algebras. The question arises as to whether this method can be applied to other gauge groups.

Let G be a semi-simple Lie group. Every element of the Yang–Mills group G^{∞} generated by G can be written in the form $\exp(\mathrm{i}\Sigma_{k=1}^{N} h_k(x)F_k)$ where F_k are generators of Lie algebra of G

$$[F_i, F_k] = \mathrm{i}c_{ik}{}^{j} F_j;$$

x is a point in the Minkowski space and $N = \dim G$. Expanding $h_k(x)$ in the power series we get

$$\exp\left(\mathrm{i} \sum_{k=1}^{N} h_k(x)F_x\right) = \exp\left(\mathrm{i} \sum_{k=1}^{N} \sum_{n=0}^{\infty} \sum_{\mu_1} \sum_{\mu_n} \alpha_k^{\mu_1 \ldots \mu_n} x_{\mu_1} \ldots x_{\mu_n} F_k\right)$$

where $\alpha_k^{\mu_1 \ldots \mu_n}$ are the expansion coefficients. The operators

$$F_{k,\mu_1 \ldots \mu_n} \equiv x_{\mu_1} \ldots x_{\mu_n} F_k \tag{1}$$

can be treated as the basis of the Lie algebra of the Yang–Mills group. We note that $F_{k,\mu_1 \ldots \mu_n}$ is symmetric in indices μ_j, $j = 1, 2, \ldots, n$. The commutation rules are

$$[F_{k,\mu_1 \ldots \mu_n}, F_{l,\nu_1 \ldots \nu_m}] = \mathrm{i}c_{kl}{}^{j} F_{j,\mu_1 \ldots \mu_n \nu_1 \ldots \nu_m}. \tag{2}$$

In particular equation (2) gives

$$[F_{k,\nu_0}, F_{l,\nu_1 \ldots \nu_n}] = \mathrm{i}c_{kl}{}^{j} F_{j,\nu_0 \ldots \nu_n}. \tag{3}$$

Taking into account the semi-simplicity of G ($\det[g_{ik}] = \det[-\frac{1}{2}c_{ri}{}^{s}c_{sk}{}^{r}] \neq 0$), we obtain by induction

$$F_{r,\nu_0 \ldots \nu_n} = (-\tfrac{1}{2}i)^n c_r^{k_1 i_1} c_{i_1}^{k_2 i_2} c_{i_2}^{k_3 i_3} \ldots c_{i_{n-1}}^{k_n i_n} [F_{k_1,\nu_1}[F_{k_2,\nu_2}[\ldots [F_{k_n,\nu_n}, F_{i_n,\nu_0}] \ldots]. \tag{4}$$

Of course, in an abstract (realization-independent) case the right-hand side should be symmetrized in indices ν_k. Denoting by P_μ ($=i\partial_\mu$) the translations operators, we obtain

$$[\ldots[F_{k,\nu_1\ldots\nu_n}, P^{\nu_1}]P^{\nu_2}]\ldots P^{\nu_n}] = (-i)^n \frac{(n+3)!}{3!} F_k. \tag{5}$$

In an abstract case, if F_k and $F_{k,\mu}$ differs from zero simultaneously it follows from equations (4), (5) that $F_{k,\nu_1\ldots\nu_n} \neq 0$ for every n. Moreover, equation (5) implies that for $m \neq n$ the generators $F_{k,\mu_1\ldots\mu_m}$ and $F_{l,\nu_1\ldots\nu_n}$ are linearly independent. Taking into account Lorentz covariance of $F_{k,\mu_1\ldots\mu_n}$ we see that the same is true for $m=n$, $(\mu_1\mu_2\ldots\mu_n) \neq (\nu_1\nu_2\ldots\nu_n)$. From the commutation rules in the Cartan basis of the algebra of G

$$\begin{aligned} &[H_r, H_s] = 0, \qquad [H_r, E_\alpha] = \phi_r(\alpha)E_\alpha, \\ &[E_\alpha, E_{-\alpha}] = \phi_r(\alpha)H_r, \qquad [E_\alpha, E_\beta] = N_{\alpha\beta}E_{\alpha+\beta}, \qquad \beta \neq -\alpha \end{aligned} \tag{6}$$

we obtain

$$[H_{r,\mu}, H_{l,\nu}] = 0 \tag{7a}$$

$$[H_{r,\mu}, H_l] = 0 \tag{7b}$$

$$[H_{r,\mu}, E_\alpha] = \phi_r(\alpha)E_{\alpha,\mu} \tag{7c}$$

and because of $[P_\mu, E_{\alpha,\nu}] = ig_{\mu\nu}E_\alpha$ it follows that $E_{\alpha,\mu} \neq 0$ if $E_\alpha \neq 0$. So we have the following theorem.

The Lie algebra of the Yang–Mills group† generated by the semi-simple Lie group with the Lie algebra spanned by the generators $\{H_k, E_\alpha\}$ (or $\{F_j\}$) is the algebraic closure of the Lie algebras

$$\{H_k, E_\alpha\} \oplus \{P_\mu, J_{\mu\nu}\} \tag{8a}$$

and

$$\{H_k, H_{k,\mu}, P_\mu, J_{\mu\nu}\} \tag{8b}$$

where the last one is the extension of the Poincaré algebra by Abelian algebra defined by equations (7a) and (7b), and

$$[H_k, P_\mu] = 0, \qquad [H_{k,\nu}, P_\mu] = -ig_{\mu\nu}H_k;$$

$\{H_k\}$ is simultaneously the Cartan subalgebra of $\{H_k, E_\alpha\}$. Of course, the algebras defined by equations (8a) and (8b) have the common subalgebra $\{H_k\} \oplus \{P_\mu, J_{\mu\nu}\}$. Every theory which is invariant under the groups generated by these algebras is invariant under the Yang–Mills group.

2. Realizations of G^∞

Let

$$\{F_j\} = A \oplus V$$

be the Cartan (symmetric) decomposition of the Lie algebra $\{F_j\}$. Then $\{H_k\} \subset V$.

† In fact we are interested in simultaneous realizations of the Yang–Mills and Poincaré groups (generated by P_μ and $J_{\mu\nu}$).

Denoting

$$L \equiv \{F_i\} \oplus \{P_\mu\}, \qquad U \equiv \{H_k\} \oplus \{H_{k,\mu}\} \oplus \{P_\mu\},$$
$$E^A \equiv \{E_\alpha\} \cap A, \qquad E^V \equiv \{E_\alpha\} \cap V$$

we have the situation illustrated in figure 1. We assume that the group $G(L)$ (generated by L) is in general realized nonlinearly (and linearizes on $G(V)$). It follows that the realization of $G(U)$ must be linear on $G(H)$, i.e. the Cartan subgroup of G. Because we assume that $\{P_\mu\}$ is realized in the standard fashion, i.e. nonlinearly, we see from the commutation rule $[P_\mu, H_{k,\nu}] = \mathrm{i}g_{\mu\nu}H_k$ that $\{H_{k,\mu}\}$ must be realized nonlinearly too.

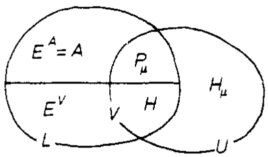

Figure 1. The Cartan decomposition of the algebras L and U.

We follow the standard procedure for finding nonlinear realizations for both groups $G(L)$ and $G(U)$. After doing this we choose only those realizations which are identical on the common subgroup of $G(L)$ and $G(U)$, i.e. $G(H \oplus P)$.

2.1. *The group G(L)*

Because of the commutation rule $[P_\mu, F_j] = 0$ we can choose the parametrization of the coset space $G(L)/G(V)$ in the form $\mathrm{e}^{\mathrm{i}xP} \exp(\mathrm{i}\phi_\alpha E^A_\alpha)$. So we have

$$G(L) \ni g: \qquad g(\mathrm{e}^{\mathrm{i}xP}\mathrm{e}^{\mathrm{i}\phi E A}) = \mathrm{e}^{\mathrm{i}(x+a)P}\mathrm{e}^{\mathrm{i}\phi'(g,\phi)EA}\,\mathrm{e}^{\mathrm{i}\theta(g,\phi)V} \tag{9}$$

i.e. $x_\mu \to x_\mu + a_\mu$, $\phi \to G(L)\phi \equiv \phi'(g, \phi)$ for preferred fields and $\psi \to \mathscr{D}(\mathrm{e}^{\mathrm{i}\theta(g,\phi)V})\psi$ for others.

2.2. *The group G(U)*

In this case there exists only one parametrization $\mathrm{e}^{\mathrm{i}xP}\exp(\mathrm{i}A_k{}^\mu H_{k\mu})$ of $G(U)/G(H)$ which is consistent with the condition $[P_\mu, F_j] = 0$. We have

$$G(U) \ni g: \qquad g[\mathrm{e}^{\mathrm{i}xP}\exp(\mathrm{i}A_k{}^\mu H_{k\mu})] = \mathrm{e}^{\mathrm{i}(x+a)P}\exp[\mathrm{i}(A_k{}^\mu + \alpha_k{}^\mu)H_{k\mu}]$$
$$\times \exp[\mathrm{i}(\alpha_k + \alpha_k{}^\mu X_\mu + \tfrac{1}{2}\alpha_k{}^\mu a_\mu)H_k] \tag{10a}$$

i.e.

$$x_\mu \to x_\mu + a_\mu,\ A_{k\mu} \to A_{k\mu} + \alpha_{k\mu},\ \psi \to \mathscr{D}'\{\exp[\mathrm{i}(\alpha_k + \alpha_k{}^\mu x_\mu + \tfrac{1}{2}\alpha_k{}^\mu a_\mu)H_k]\}\psi \tag{10b}$$

where $g = \exp[\mathrm{i}(aP + \alpha_k{}^\mu H_{k\mu} + \alpha_k H_k)]$. The consistency condition implies:

(i) on the common subgroup $G(H)$ the representations $\mathscr{D}$ and $\mathscr{D}'$ are identical, i.e. $\mathscr{D}'(H) = \mathscr{D}(V) \downarrow G(H)$;

(ii) for $g \in G(U)$ the preferred fields ϕ_α transform as follows

$$\phi \to \phi' = \Delta\{\exp[\mathrm{i}(\alpha_k + \alpha_k{}^\mu x_\mu + \tfrac{1}{2}\alpha_k{}^\mu a_\mu)H_k]\}\phi$$

where $\Delta(H)\phi = G(L)\phi \downarrow G(H)$;

(iii) the fields $A_{k\mu}$ may transform twofold under $G(L)$: (a) $A_{k\mu}$ transform trivially, i.e. $A_{k\mu} \rightarrow A_{k\mu}$, (b) $A_{k\mu}$ belong to the multiplet

$$G_{k\mu} \equiv \begin{bmatrix} A_{k\mu} \\ B_{k\mu} \end{bmatrix}$$

which transforms under $G(L)$ as follows:

$$G_\mu \rightarrow D(e^{i\theta(g,\phi)V})G_\mu$$

where D is the representation of the group $G(V)$ that on subgroup $G(H)$ has the form

$$\begin{bmatrix} I & 0 \\ 0 & D'(H) \end{bmatrix} \qquad \text{(i.e. } H_k A_{\mu i} = 0\text{)};$$

under $G(U)$ the fields $B_{k\mu}$ transform according to the rule $(10b)$, i.e.

$$B_\mu \rightarrow D'\{\exp[i(\alpha_k + \alpha_k{}^\mu x_\mu + \tfrac{1}{2}\alpha_k{}^\mu a_\mu)H_k]\}B_\mu.$$

3. Covariant derivatives

It can be shown that in the case (iii$a$) ($A_{k\mu}$ transform trivially under $G(L)$) it is impossible to construct the covariant derivatives common for both groups $G(L)$ and $G(U)$, i.e. the covariant derivatives of G^∞. In the case (iii$b$) the consistency condition implies that the multiplet G_μ transforms according to the adjoint representation of $G(V)$ and we obtain nonlinear gauge theory (in the case where $V = \{F_k\}$ is linear) investigated by Salam and Strathdee (1969), Coleman *et al* (1969) and Callan *et al* (1969).

4. Conclusions

From the above we see that the Yang–Mills theory can be considered as standard nonlinear theory. In this context we see that the (compensating) vector fields $A_{k\mu}$ must occur because they are preferred fields connected with the generators $H_{k\mu}$.

We would like to point out that not all gauge fields can be treated in the same manner—essentially nonlinear fields are only preferred fields $A_{k\mu}$.

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